Exact solution for the spatially homogeneous nonlinear Kac model of the Boltzmann equation with an external force term

1984 J. Phys. A: Math. Gen. 17 2355

(http://iopscience.iop.org/0305-4470/17/11/031)

J. Phys. A: Math. Gen. 17 (1984) 2355–2361. Printed in Great Britain

ADDENDUM

Exact solution for the spatially homogeneous nonlinear Kac model of the Boltzmann equation with an external force term

H Cornille

Service de Physique Théorique, CEN Saclay, 91191 Gif-sur-Yvette Cedex, France

Received 29 February 1984

Abstract. We study the spatially homogeneous Kac model of the nonlinear Boltzmann equation in $1+1$ dimensions (velocity v and time t) when an external force term is present. We find a closed solution. The force term decreases exponentially with time, like constant$_1$ exp $-$ constant$_2 \times t$, where both constants depend explicitly on the moments of the cross section. We find, for the relaxation to equilibrium, that the Tjon overpopulation effect depends on both the initial condition and the microscopic model of cross section. For the existence of this effect, we establish a criterion which is a well defined linear combination of the moments of the cross section.

The full Kac model (Kac 1956, Uhlenbeck and Ford 1963), which is a nonlinear Boltzmann equation, depends on the three variables v, t, x (velocity v, time t, position x) and the distribution function $f(v, t, x)$ satisfies the equation

$$(\partial_t + v\partial_x + \Lambda(t, x)\partial_v)f(v) = \int_{-\pi}^{+\pi} \sigma(\theta) \int_{-\infty}^{+\infty} (f(v')f(w') - f(v)f(w))\, \mathrm{d}w\, \mathrm{d}\theta,$$
$$v' = v\cos\theta - w\sin\theta, \qquad w' = v\sin\theta + w\cos\theta, \qquad \sigma(\theta) = \sigma(-\theta), \tag{1}$$

where $f(v)$ means $f(v, t, x)$ and $\sigma(\theta)$ is the scattering cross section. If the gradient term $v\partial_x$ is not present $f = f(v, t)$ is the spatially homogeneous solution with the force term $\Lambda = \Lambda(t)$. When this force term is absent, we have recently found (Cornille 1984a) a solution in closed form

$$f(v, t) = \left(\frac{b}{2\pi}\right)^{1/2} \mathrm{e}^{-bv^2/2}\left(\frac{3-b}{2} + \frac{dbv}{\sqrt{2}}\, \mathrm{e}^{-(\tau_0-\tau_1)t} + b(b-1)\frac{v^2}{2}\right), \qquad b = (1 - c\, \mathrm{e}^{-\sigma_2 t})^{-1},$$
$$\tau_m = \int_{-\pi}^{+\pi} (\cos\theta)^m \sigma(\theta)\, \mathrm{d}\theta, \qquad \tau_0 = 1, \qquad \sigma_2 = \tau_2 - \tau_4, \qquad \sigma_2 - \tau_1 + \tau_3 = 0, \tag{2}$$

where the even velocity part $f^+(v, t)$ is the BKW even mode (Bobylev 1975, Krook and Wu 1976). This even mode has been obtained by Ernst (1979, 1980, 1981) for the Kac model and the odd velocity part $f^-(v, t)$ in (2) ($f = f^+ + f^-$) is an associated nontrivial odd mode.

For this complete BKW solution (2), very interesting new results were obtained for the overpopulation Tjon (1979) effect of high velocity particles' relaxation to equilibrium. Define a phenomenological criterion

$$\mathrm{crit}_1 = \tau_0 - 2\sigma_2 - \tau_1 \tag{3}$$

as a linear combination of $\sigma(\theta)$ moments. Roughly speaking, the Tjon effect exists (or not) depending on whether crit_1 is negative (or not). If further, still for the spatially homogeneous Kac model without external force, we investigate the general odd solutions associated to the BKW even mode, then this criterion is still a good tool in the study of the Tjon effect (Cornille 1984a, b).

The aim of this paper is to introduce an external force term $\Lambda(t)\partial_v$ into the spatially homogeneous Kac model and try to find, if it exists, the generalisation of the closed solution (2) and investigate the Tjon effect for this enlarged model.

Different methods exist for the search for closed solutions. One can either try the direct substitution of an appropriate ansatz or study the differential system for the Laguerre moments of $f(v, t)$. Kac's equations for $f(v, t)$ are

$$\partial_t f^+ + \Lambda(t)\partial_v f^- = \int_{-\pi}^{+\pi} \sigma(\theta) \int_{-\infty}^{+\infty} (f^+(v', t)f^+(w', t) - f^+(v, t)f^+(w, t))\, \mathrm{d}w\, \mathrm{d}\theta, \tag{4a}$$

$$\partial_t f^- + \Lambda(t)\partial_v f^+ = \int_{-\pi}^{+\pi} \sigma(\theta) \int_{-\infty}^{+\infty} (f^-(v', t)f^+(w', t) - f^-(v, t)f^+(w, t))\, \mathrm{d}w\, \mathrm{d}\theta. \tag{4b}$$

In the first method, assuming

$$\sqrt{2\pi} f^\pm(v, t)\, \mathrm{e}^{b(t)v^2/2} = \sum_0^{n_\pm} \alpha_{2n+(1\mp1)/2}(t) v^{2n+(1\mp1)/2}$$

and substituting into ($4a$, b), we find both that the only possibility is $n_+ = 1$, $n_- = 0$ and these equations become v^{2m}, v^{2m+1} polynomials such that the time dependent coefficients are zero. In ($4a$) let us write down the coefficient of v^4 and, for a reason which will become transparent below, the sums of the coefficients of $v^2/2$, $2bv^0$ and $3v^2/4$, bv^0:

$$\begin{aligned} &\alpha_2 = \frac{-b^{1/2}}{\sigma_2} b_t, \qquad 1 = \alpha_0 b^{-1/2} + \frac{\alpha_2}{2} b^{-3/2}, \\ &\partial_t\left(\frac{1}{\sigma_2}\partial_t(b^{-1}) + b^{-1}\right) = \sqrt{2}\Lambda\alpha_1 b^{-3/2}. \end{aligned} \tag{5a}$$

The last two relations in ($5a$) correspond to conserved quantities. If in ($4a$) we multiply by the invariants 1, v^2 and integrate over v, then, as is well known, the collision term contributions vanish. We find $\int_{-\infty}^{+\infty} f\, \mathrm{d}v = \text{constant}$, setting this constant equal to 1 in the second ($5a$) relation and $\partial_t \int_{-\infty}^{+\infty} v^2 f^+\, \mathrm{d}v = 2\Lambda \int_{-\infty}^{+\infty} v f^-\, \mathrm{d}v$ written down in the last ($5a$) relation. In ($4b$) we find two terms proportional to v and v^3. Taking into account ($5a$) for the corresponding coefficients, we choose the v^3 relation and another one which is the difference of the coefficients of $v/\sqrt{2}$, $-\frac{3}{2}(v/\sqrt{2})^3$ that we can directly integrate. We find

$$\begin{aligned} &\Lambda(t) = (\lambda_\mathrm{F}/\sqrt{2}) d\, \mathrm{e}^{-t\tau_\mathrm{F}}, \qquad \alpha_1 = b^{3/2} d\, \mathrm{e}^{-t\tau_\mathrm{F}}, \\ &\lambda_\mathrm{F} = \sigma_2 - \tau_1 + \tau_3, \qquad \tau_\mathrm{F} = \tau_0 - \sigma_2 - \tau_3 > 0, \end{aligned} \tag{5b}$$

where $\tau_\mathrm{F} = -\int \sigma(\theta)(1 - \cos\theta)(1 + \cos\theta - \cos^3\theta)\, \mathrm{d}\theta > 0$, $\lambda_\mathrm{F} = -\int \sigma(\theta) \cos(1 - \cos\theta)^2 \times (1 + \cos\theta)\, \mathrm{d}\theta$ has no definite sign and d is an arbitrary constant. In order to determine $b(t)$ we substitute the relations ($5b$) into ($5a$) and obtain a non-homogeneous second-order differential equation $\partial_t(\sigma_2^{-1}\partial_t b^{-1} + b^{-1}) = d^2\lambda_\mathrm{F}\, \mathrm{e}^{-2t\tau_\mathrm{F}}$. The integration gives us two supplementary arbitrary parameters $b^{-1} = c_0 - c\, \mathrm{e}^{-\sigma_2 t} + c_1\, \mathrm{e}^{-2t\tau_\mathrm{F}}$ where c_1 depends on both d and the moments of $\sigma(\theta)$ and $c_0 = 1$ if we require $f(v, \infty) = \exp(-v^2/2)$. Finally our closed (spatially homogeneous with external force term) solution reads

$$f=\frac{e^{-bv^2/2}}{\sqrt{2\pi}}\sqrt{b}\left[\frac{3-b}{2}+\frac{b}{2}e^{-2\tau_F t}\frac{d^2\lambda_F}{2\tau_F}+\frac{v}{\sqrt{2}}bd\,e^{-\tau_F t}+\frac{v^2}{2}b\left(b-1-b\,e^{-2\tau_F t}\frac{d^2\lambda_F}{2\tau_F}\right)\right],$$
$$b^{-1}=1-c\,e^{-\sigma_2 t}+c_F\,e^{-2\tau_F t},\qquad c_F=\frac{\sigma_2 d^2\lambda_F}{2\tau_F(2\tau_F-\sigma_2)},\qquad \Lambda=\frac{d\lambda_F}{\sqrt{2}}e^{-\tau_F t},\tag{6}$$

λ_F, $\tau_F>0$, being defined in (5*b*), c and d being arbitrary constants, $2\tau_F-\sigma_2=\int\sigma(\theta)(1-\cos\theta)^2(2+4\cos\theta+3\cos^2\theta)>0$ and $b^{-1}\simeq 1-ce^{-\sigma_2 t}$ for t large.

Let us compare the two exact solutions (2) and (6) when an external force is present. If $\lambda_F=\sigma_2-\tau_1+\tau_3=0$, then the force term disappears and (6) reduces to the complete BKW solution (2). When $\lambda_F\neq 0$, the solution (6) can be seen as a generalisation of the BKW solution with an external force present. We notice that the even part $f^+(v,t)$ in (6) is different from the corresponding BKW even mode in (2) because it has in the square brackets a supplementary factor $\frac{1}{2}b\,e^{-2\tau_F t}d^2\lambda_F/2\tau_F$ proportional to d, the integration constant of the odd part $f^-(v,t)$. At $t=0$ in (6), due to the presence of λ_F, τ_F, c_F in $f(v,0)$, the initial conditions depend not only on c, $|d|$ but also on the moments of $\sigma(\theta)$. On the contrary, in (2), only c, $|d|$ enter into $f(v,0)$. Another important difference exists if we assume $\sigma(\theta)=\sigma(\pi-\theta)$ leading to $\tau_{2m+1}=0$. In (2), the condition $\sigma_2-\tau_1+\tau_3=0$ becomes $\sigma_2=0$ which is impossible for $\sigma(\theta)>0$ and the solution does not exist. On the contrary in (6) this symmetry $\sigma(\theta)=\sigma(\pi-\theta)$ leads to $\lambda_F=\sigma_2$, $\tau_F=\tau_0-\sigma_2$ and the solution still exists but $f^-(v,t)$ is not a trivial solution as it is the case when the external force is absent.

Let us now discuss the positivity constraint $f(v,0)>0$ at $t=0$. We must have $b(0)>0$, $\alpha_0(0)>0$, $\alpha_2(0)>0$ or

$$\sigma_2^{-1}2\tau_F c_F<c<\inf(1+c_F,\tfrac{2}{3}+\tfrac{2}{3}c_F[(\sigma_2+\tau_F)/\sigma_2]),\tag{7}$$

and the discriminant of the second-order polynomial in v in the square brackets of (6) must be negative:

$$d^2+[2/(1-c+c_F)](2-3c+c_F[(\sigma_2+\tau_F)/\sigma_2])(-c+2\tau_F c_F/\sigma_2)<0.\tag{8}$$

We explicitly see that these constraints depend on both c, $|d|$ and the moments of $\sigma(\theta)$. This means that we must introduce models for $\sigma(\theta)$ into the discussion, and for simplicity we choose the simplest one

$$\sigma(\theta)=\tfrac{1}{2}(\delta(\theta-\theta_1)+\delta(\theta+\theta_1)),\qquad 0<|\cos\theta_1|<1.\tag{9}$$

The positivity of $f(v,0)$ is satisfied in a domain of the c, $|d|$, $\cos\theta_1$ space. In figure 1 we plot the region of the c, $|d|$ plane where for $\sigma(\theta)$ given by (9) we have found $\cos\theta_1$ values (not necessarily all $\cos\theta_1$ values) such that $f(v,0)>0$. The broken line in figure 1 corresponds to $|d|^2(1-c)=2c(2-3c)$ where $f(v,0)>0$ for the solution (2). (In that (2) case, the domain does not explicitly depend on $\sigma(\theta)$ although for instance for $\sigma(\theta)$ given by (9) no solution exists.)

Another method in the search for closed solutions of the Boltzmann equation is the study of the Laguerre expansion. For the spatially homogeneous Kac model, without external force, Kac (1956) and Ernst (1980) have obtained the equations for the Hermite moments. The equations for the Laguerre moments with the gradient term $v\partial_x$ were recently given (Cornille 1984a, b). Here we add the force term $\Lambda(x,t)\,\partial f/\partial v$ and take great advantage of the relations for the Laguerre polynomials $(1-2y+$

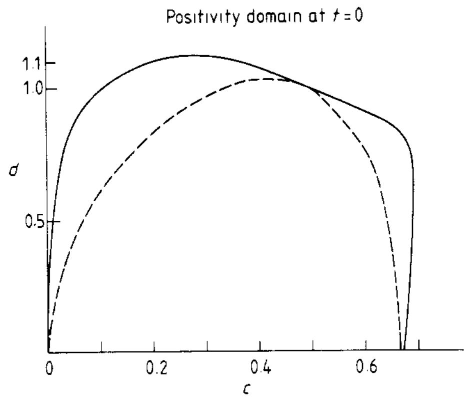

Figure 1. $|d|$, c region where $f(v, 0) > 0$ for (6) and $\sigma(\theta)$ given by (9). Broken line: positivity domain for (2).

$2y\,\partial y)L_n^{(1/2)}(y) = 2(n+1)L_{n+1}^{(-1/2)}(y)$, $L_n^{-1/2} = L_n^{1/2} - L_{n-1}^{1/2}$, $\partial_y L_n^{(-1/2)}(y) = -L_{n-1}^{(1/2)}(y)$. We define

$$\begin{aligned} f^{+}\sqrt{2\pi}\,\mathrm{e}^{v^2/2} &= \sum L_n^{-1/2}(v^2/2) D_n^{+}(x, t)(-1)^n, \\ f^{-}\sqrt{2\pi}\,\mathrm{e}^{v^2/2} &= (v/\sqrt{2}) \sum L_n^{1/2}(v^2/2)(-1)^n D_n^{-}(x, t), \end{aligned} \tag{10}$$

substitute into (1) and formally find for the Laguerre moments

$$\begin{gathered} \partial_t D_n^{+} + \sqrt{2}\partial_x[(n+\tfrac{1}{2})D_n^{-} + nD_{n-1}^{-}] - \Lambda\sqrt{2}nD_{n-1}^{-} = \sum_0^n D_q^{+} D_{n-q}^{+} C_n^q B_{qn}, \\ B_{0n} = \tau_{2n} - \tau_0, \qquad B_{qn} = \sum_0^q (-1)^m C_q^m \tau_{2(n+m-q)}, \end{gathered} \tag{11a}$$

$$\begin{gathered} \partial_t D_n^{-} + \sqrt{2}\partial_x(D_n^{+} + D_{n+1}^{+}) - \Lambda\sqrt{2}D_n^{+} = \sum_0^n D_q^{+} D_{n-q}^{-} C_n^q E_{qn}, \\ E_{0n} = \tau_{2n+1} - \tau_0, \qquad E_{qn} = \sum_0^q (-1)^m C_q^m \tau_{2(n-m+q)+1}. \end{gathered} \tag{11b}$$

In the following we return to the spatially homogeneous case $f(v, t)$ with $\Lambda = \Lambda(t)$. The reader can verify that the system (11a, b) for the Laguerre moments has the solution $\Lambda(t) = (d/\sqrt{2})\lambda_{\mathrm{F}}\,\mathrm{e}^{-\tau_{\mathrm{F}}t}$ and

$$\begin{gathered} D_n^{+} = (-1)^n[w^n(1-n) + nw^{n-1}\,\mathrm{e}^{-2\tau_{\mathrm{F}}t} d^2\lambda_{\mathrm{F}}/2\tau_{\mathrm{F}}], \qquad D_n^{-} = (-1)^n w^n d\,\mathrm{e}^{-\tau_{\mathrm{F}}t}, \\ w = c\,\mathrm{e}^{-\sigma_2 t} - c_{\mathrm{F}}\,\mathrm{e}^{-2\tau_{\mathrm{F}}t}. \end{gathered} \tag{6$'$}$$

With the help of the generating functionals for the $L_n^{\pm 1/2}$ and some trivial algebra, the substitution of (6′) into (10) leads to the solution (6).

Now we discuss the Tjon overpopulation effect for the exact solution (6). We define the reduced distribution function $F(v, t) = f(v, t)/f(v, \infty)$ and study the relaxation to equilibrium $F \to 1$ for $t \to \infty$ and velocities larger in modulus than the one present at $t = 0$. Either $F \to 1$ in a monotonic way from below (no effect) or there exists high velocity for which, at intermediate times, F is substantially larger than 1 (effect).

Let us discuss numerical results obtained with the simple model for $\sigma(\theta)$ given by (9). Setting in figure 2 $c = 0.5$, $d = -0.775$, $\cos\theta_1 = 0.9$, we see the Tjon effect; $f(v, 0)$

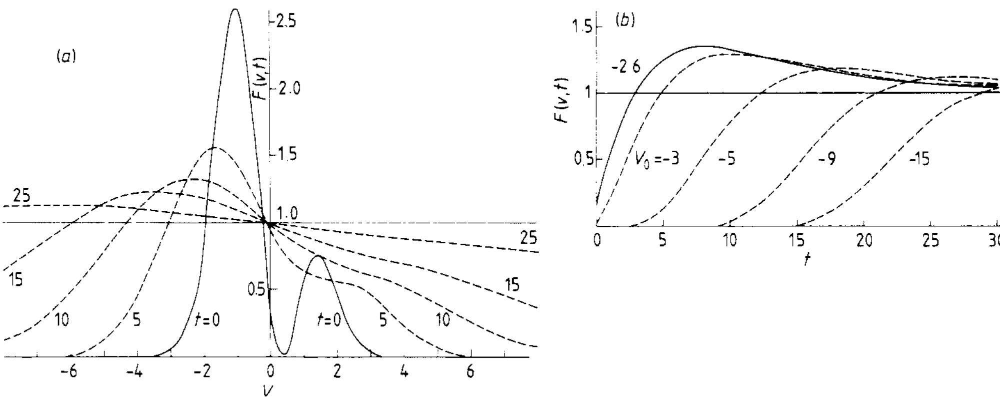

Figure 2. (*a*) Plot of $F(v, t)$ against v for $c = 0.5$, $d = -0.775$, $\cos\theta_1 = 0.9$ in (9), crit $= -0.19$. (*b*) as (*a*) but plot of $F(v, t)$ against t.

has a narrow peak, the last zero of $F-1$ moves toward $-\infty$ when t increases and the values where $F > 1$ are important. In figure 3 for $c = 0.5$, $d = -0.9$, $\cos\theta_1 = 0.5$, $f(v, 0)$ has still a narrow peak, the last zero of $F-1$ does not move when t increases and there is no effect. In figure 4 for $c = 0.5$, $d = -0.07$, $\cos\theta_1 = 0.9$, $f(v, 0)$ has two wide bumps, the zero of $F-1$ is moving but the values where $F > 1$ are very close to 1 and we conclude that the effect does not exist. These examples are generic of classes of solutions. Let us choose for c, $|d|$ values such that $f(v, 0)$ has a narrow peak (or not) and let $\cos\theta_1$ vary between -1 and $+1$. Concerning the last zero of $F-1$ we observe a transition: for $\cos\theta_1 < 0.6$ this zero does not move, for $0.7 < \cos\theta_1 < 0.93$ it moves and for $\cos\theta_1 < 0.95$ the displacement of the zero disappears. If we have a narrow peak (or not) at $t = 0$ then we observe the Tjon effect like in figure 2 (or not, like in figure 4) for $0.7 < \cos\theta_1 < 0.93$ and no effect like in figure 3 for $\cos\theta_1$ outside this interval.

Secondly, we discuss the effect in a semi-theoretical way, extending the Hauge and Praestgaard (1981) arguments for even velocity distribution alone. Let us retain for the solution (9) or (10)–(9′) the contribution coming from the first even and odd Laguerre moments and replace $L_n^{\mp 1/2}(v^2/2)$ by their dominant parts when $|v|$ is large. If we write $F - 1 \simeq |v|\, e^{-\tau_F t}[\;\;]$ then the bracket has four terms. The first one, coming from D_0^- is a constant, the second and the third, from $D_1^{\mp}$, are of the type $|v|^{1+(1\pm 1)/2}$ multiplied

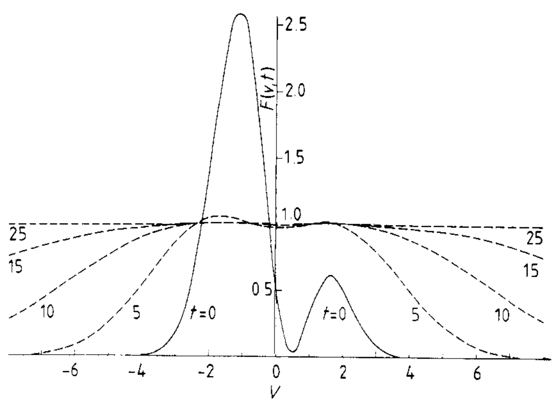

Figure 3. As figure 2(*a*) but $d = -0.9$, $\cos\theta_1 = 0.5$, crit $= 0.31$.

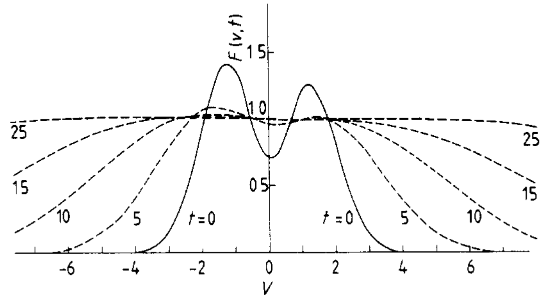

Figure 4. As figure 2(a) but $d=-0.07$, crit $=-0.19$.

by $e^{-\sigma_2 t}\, e^{-\tau_F t}$, exponential time decreasing terms, and the last one coming from D_2^+ has a time dependence which increases or decreases following the $\sigma(\theta)$ model. We shall compare the first and the last term and define a criterion:

$$F-1 \simeq |v|\, e^{-\tau_F t}[d\, \mathrm{sgn}(v)/\sqrt{2}-\tfrac{1}{8}|v|^3 c^2\, e^{\mathrm{crit}\times t}], \qquad \mathrm{crit}=\tau_F-2\sigma_2,$$
$$\mathrm{crit}=\tau_0-3\sigma_2-\tau_3=\int \sigma(\theta)(1-Z)(1+Z-2Z^2-3Z^3)\,\mathrm{d}\theta, \qquad Z=\cos\theta, \tag{12}$$

which has no definite sign. From (12) if crit >0 then for t and $|v|$ large, $F<1$ and the Tjon effect does not exist. On the contrary if for $\sigma(\theta)$ models we have crit <0 then, for $|v|$ large, the bracket in (12) has a zero moving and the effect can exist. In order to test this criterion let us return to the family of cross sections $\sigma(\theta)$ given by (9). In this case crit $=(1-Z_1)(1+Z_1-2Z_1^2-3Z_1^3)$, $Z_1=\cos\theta_1$, becomes negative for $Z_1>0.65$ and we have numerically verified (see the above discussion) that the displacement of the zero occurs near this value. Some remarks are in order. First, this criterion, which only depends on $\sigma(\theta)$, is concerned with the displacement of the zero, while the existence of the effect requires also particular initial conditions (existence or not of a narrow peak). Second, a finer analysis must also include the contribution coming from the Laguerre moments $D_1^{\mp}$ which were neglected (for instance when crit <0 is large in modulus, the four terms of the bracket have all exponential time decreasing factors, requiring an analysis outside the scope of this paper). Third if $\sigma_2=\tau_1-\tau_3>0$, the two solutions (2), (6) as well as the two criteria (3), (12) coincide, showing that (6) is indeed the generalisation of (2) for those $\sigma(\theta)\neq\sigma(\pi-\theta)$ for which τ_{2m+1} are different from zero.

We recall that for the determination of the solution (6) we never needed the assumption $\sigma(\theta)\neq\sigma(\pi-\theta)$. On the contrary, when the external force term is absent and for the non-trivial odd part $f^-(v,t)$ then this constraint was essential (Cornille 1984a, b).

Here, the exact solution (6) still exists if we assume the special symmetry $\sigma(\theta)=\sigma(\pi-\theta)$ or $\tau_{2m+1}=0$. In that case we have in (6), $\lambda_F=\sigma_2$, $\tau_F=\tau_0-\sigma_2$ and in (12), crit $=\tau_0-3\sigma_2$. For instance, the simplest case is the isotropic one $\sigma(\theta)=(2\pi)^{-1}$ for which crit $=0.625>0$. In figure 5 we plot the solution for $c=0.5$, $d=-0.8$ and we see that, in accordance with our criterion, no Tjon effect exists.

We can intuitively understand the displacement of the zero in $F-1$ with simple arguments. When both t and $|v|$ become large, the dominant terms in the odd and even parts have both different asymptotic velocity behaviours v, $|v|^4$ or $|v|$ (sgn v, $|v^3|$)

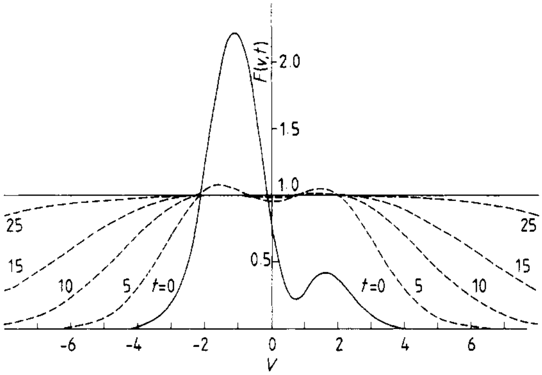

Figure 5. As figure 2(a) but $\sigma(\theta)=(2\pi)^{-1}$, $d=-0.8$, crit $=0.625$.

and different relaxation times τ_F^{-1} and $(2\sigma_2)^{-1}$. If the difference between these two relaxation times, which is proportional to crit $=\tau_F-2\sigma_2$, is positive then $F-1$ has the sign of the even part (negative here for (6)), whereas if it is positive, then when $|v|$ and t increase, there always exist values for which even and odd parts are comparable and vanish.

It remains in the future to select those of the properties, found here for the closed solution (6), which are general and for that purpose to study the system of equations for the Laguerre moments, written down in (11a, b).

Acknowledgment

I thank Professor E Brézin for discussions.

References

Bobylev V 1975 *Dokl. Akad. Nauk* **225** 1296
—— 1976 *Sov. Phys.-Dokl.* **20** 823
Cornille H 1984a *J. Phys. A: Math. Gen.* **17** L235
—— 1984b *Nonlinear Kac's model: I spatially homogeneous solutions and the Tjon effect, Saclay preprint 84-15*
Ernst M H 1979 *Phys. Lett.* **69A** 390
—— 1980 *Fundamental problems in statistical mechanics V* ed E G D Cohen (Amsterdam: North-Holland) p 249
—— 1981 *Phys. Rep.* **78** 1
Hauge E H and Praestgaard E 1981 *J. Stat. Phys.* **24** 21
Kac M 1956 *Proc. 3rd Berkeley Symp. on Mathematics Statistics and Probability* vol 3, p 111 (Berkeley: University of California)
Krook M and Wu T T 1976 *Phys. Rev. Lett.* **16** 1107
Tjon J A 1979 *Phys. Lett.* **70A** 369
Uhlenbeck G E and Ford C W 1963 *Lectures in Statistical Mechanics* ed M Kac (Providence, Rhode Island: American Mathematical Society) pp 99–101